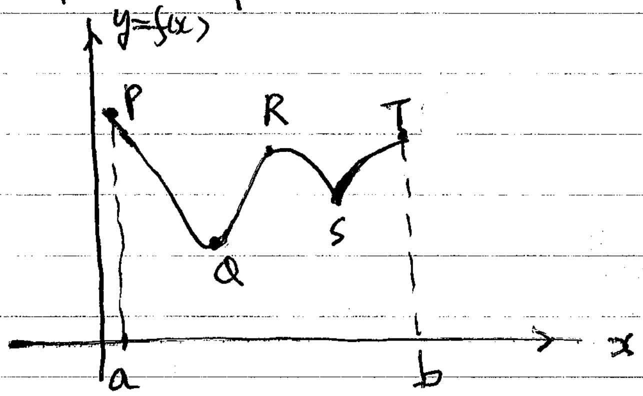
Optimization of functions of several variables:

Recall function of one variable case: $y=f(x)$, $a\le x\le b$.



P = Absolute/Global maximum (the highest pt.).

Q = Absolute/Global minimum (" lowest pt.).

S = local minimum

R, T = local maximum

Definition: For any point $c\in(a,b)$, it is said to be critical point of f iff either (i) $f'(c)=0$ or (ii) $f'(c)$ does not exist.

Th^m (Fundamental theorem for optimization) Let f be a continuous function over (a,b), we have

(i) If there is a local max or local min. at $c\in(a,b)$, then c is a critical point of f.

(ii) If f is continuous on $[a,b]$, then f attains an absolute maximum and an absolute minimum over $[a,b]$. Further, the abs. max. and abs. min. are being attained at either a critical point inside (a,b) or at the end pts.

Consider now the case of function of 2 variables:

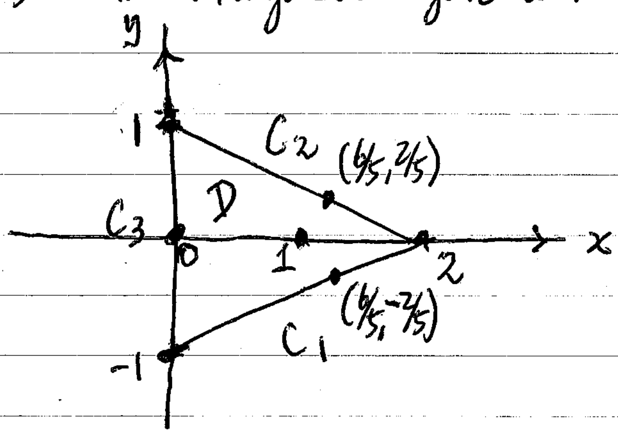
$$z=f(x,y),\quad (x,y)\in D$$

D is the domain of f, an open connected set in the xy plane with boundary ∂D. Exactly the same theory could be carried over.

Def'n. Given f continuous over D, $(a,b)\in D$ is said to be a critical point of f either (i) $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$ or (ii) one of the $\frac{\partial f}{\partial x}(a,b)$, $\frac{\partial f}{\partial y}(a,b)$ fails to exist.

Theorem: Given $z=f(x,y)$ a continuous function over D, if f attains a local minimum or a local maximum at $(a,b)\in D$, then (a,b) is a critical point of f. Further, if D is bounded and f is continuous over $\overline{D} = D\cup\partial D$ (i.e. f is continuous up to ∂D the boundary of D), then f attains an absolute max. and an absolute min. over $\overline{D}$, which is either a critical point or a boundary point.

Ex. Find the absolute max and the absolute min. of $f(x,y) = x^2+y^2-2x$, $(x,y)\in D$ where D is the triangular region as shown below.



Solution:

$f(x,y)$ is a polynomial in x & y, so it is continuous, further D is a bounded domain, therefore, the hypotheses of the theorem are satisfied.

Let us first figure out all the critical points in D by setting

$\frac{\partial f}{\partial x} = 2x-2=0$ and $\frac{\partial f}{\partial y} = 2y=0 \Rightarrow (1,0)$ is a critical pt.

Since the abs. max. and abs. min. could also be attained at ∂D, we need to find out the max & min along C_1, C_2 and C_3.

Along C_1, $y=\frac{x}{2}-1$, $0\le x\le 2 \Rightarrow f(x,y) = x^2+(\frac{x}{2}-1)^2-2x = \frac{5}{4}x^2-3x+1$, $0\le x\le 2$.

Setting $f'(x) = \frac{5}{2}x-3=0 \Rightarrow x=6/5$ $\therefore (6/5, -2/5)$ is a critical pt. along C_1.

Along C_2, $y=-\frac{x}{2}+1$, $0\le x\le 2 \Rightarrow (6/5, 2/5)$ is a critical point along C_2.

Along C_3, $x=0$, $f(x,y)=y^2$, $-1\le y\le 1 \Rightarrow (0,0)$ is a critical pt. along C_3.

Finally, we must also take into consideration the end points of C_1, C_2 & C_3 which amounts to the corner points $(0,\pm 1), (2,0)$. Thus, there are all-together 7 points where the abs. max & abs. min of f could be attained.

Evaluating f at those pts,

$f(1,0) = -1$, $f(6/5, \pm 2/5) = -4/5$, $f(0,0) = 0$, $f(0,\pm 1) = 1$ and $f(2,0) = 0$.

Therefore $f(1,0) = -1$ is the abs. min. and $f(0,\pm 1) = 1$ are abs. max. //

Ex. (D is unbounded) Find the abs. maximum of $z = f(x,y) = \frac{8x^3}{3} + 4y^3 - x^4 - y^4$

<u>Solution:</u>

First we observe that as $x, y \to \pm\infty$, $f(x,y) \to -\infty$ $\therefore$ there is no absolute minimum. Further, for (x,y) in the 1^{st} quadrant which is near $(0,0)$, $f(x,y)$ is positive, hence there must be an absolute max. which is positive. At the abs. max, we have

$$\begin{cases} \frac{\partial f}{\partial x} = 8x^2 - 4x^3 = 0 \Rightarrow x = 0, 2 \\ \frac{\partial f}{\partial y} = 12y^2 - 4y^3 = 0 \Rightarrow y = 0, 3 \end{cases}$$

(note that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ always exist and there is no critical points of the 2^{nd} type).

Thus, $(0,0)$, $(0,3)$, $(2,0)$ and $(2,3)$ are the critical points where the abs. max of f would be attained. Evaluating $f(x,y)$ at these pts.

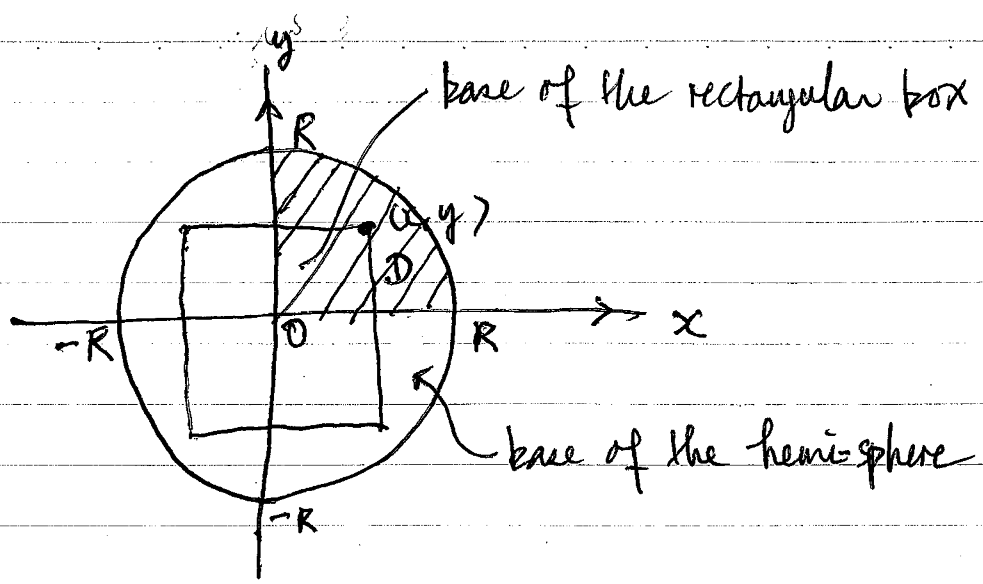
$f(0,0) = 0$, $f(2,0) = 16/3$, $f(0,3) = 27$, $f(2,3) = \frac{97}{3}$

Thus, $f(2,3) = \frac{97}{3}$ is our absolute maximum //.

Ex. What is the largest volume of a rectangular box that could be inscribed in a hemisphere of radius R? We assume one face of the box is on the base of the base of the hemisphere.

<u>Solution</u>

Without loss of generality, we may assume the center of the rectangular base of the box is at the origin of the x-y plane (see figure). Further, we let (x,y) be the corner of the base in the 1^{st} quadrant.



We note that the equation of the hemi-spherical surface is $z=\sqrt{R^2-x^2-y^2}$, which is also the height of the rectangular box, the volume of the box is therefore,

$$f(x,y)=4xy\sqrt{R^2-x^2-y^2}\quad,(x,y)\in D$$

where D is the quarter circular region of radius R in the 1^{st} quadrant. Since f is continuous over D, f must attain its absolute max and its abs. minimum over D. Further $f(x,y)=0$ along ∂D, the abs. max. must be attained at the interior of D where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist. Thus, at the point (x,y) where the absolute maximum of f is attained, we have

$$\frac{\partial f}{\partial x}=4y\sqrt{R^2-x^2-y^2}-\frac{4y^2}{\sqrt{R^2-x^2-y^2}}=\frac{4y(R^2-y^2-2x^2)}{\sqrt{R^2-x^2-y^2}}=0$$

$$\Rightarrow y^2+2x^2=R^2 \text{ ———— (1)}$$ ($y=0$ could be rejected as mentioned before).

Similary, we also have $\frac{\partial f}{\partial y}=0$ implying

$$x^2+2y^2=R^2 \text{ ———— (2)}$$

Solving (1) and (2) simultaneously, we have $x=y=\frac{R}{\sqrt{3}}$ which must be the point where the abs. max. of $f(x,y)$ is being attained (note that such result could actually be expected by symmetry argument).

The corresponding max. capacity of the box is therefore,

$$f\left(\frac{R}{\sqrt{3}},\frac{R}{\sqrt{3}}\right)=4\left(\frac{R}{\sqrt{3}}\right)^2\sqrt{R^2-\frac{2}{3}R^2}=\frac{4}{3}R^2\frac{R}{\sqrt{3}}=\frac{4R^3}{3\sqrt{3}}$$ //.

2nd derivative test for local max. and local min. when we have a smooth critical point

Thm. (2nd Derivative Test) Given $z=f(x,y)$, $(x,y)\in D$ and let $(a,b)\in D$ be a smooth critical point (f twice continuously differentiable) where $\frac{\partial f}{\partial x}(a,b)=\frac{\partial f}{\partial y}(a,b)=0$.

We define the discriminant $D_f(a,b)$ by

$$D_f(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2,$$

then we have

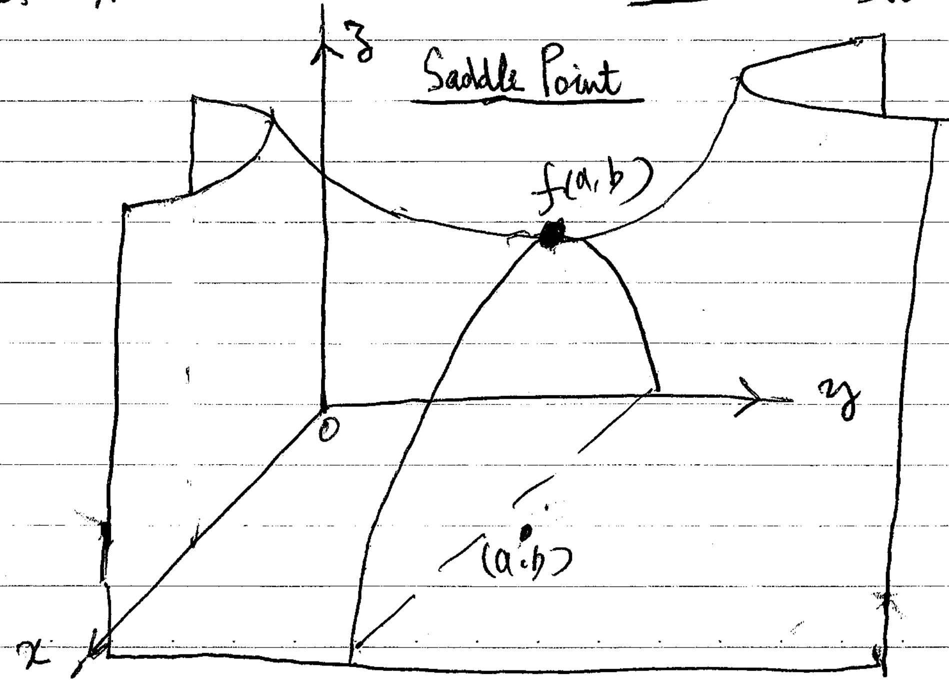
(i) If $D(a,b)>0$ and $f_{xx}(a,b)>0$, f is a local min. at (a,b)

(ii) If $D(a,b)>0$ and $f_{xx}(a,b)<0$, f is a local max at (a,b)

(iii) If $D(a,b)<0$, f has a saddle point at (a,b) (see figure below)

(iv) If $D(a,b)=0$, the test is inconclusive.

Remark: In case $D(a,b)>0$, $f_{xx}(a,b)$ and $f_{yy}(a,b)$ share the same sign. In case $D(a,b)<0$, $f_{xx}(a,b)$ and $f_{yy}(a,b)$ are opposite in sign. In that case we have a saddle as shown



$f_{xx}(a,b)>0$

$f_{yy}(a,b)<0$

Proof: Postpone until later.

Ex. Back to the earlier example $f(x,y) = \frac{8x^3}{3} + 4y^3 - x^4 - y^4$.

$\frac{\partial f(x,y)}{\partial x} = 8x^2 - 4x^3 = 0$, $\frac{\partial f(x,y)}{\partial y} = 12y^2 - 4y^3 = 0$

at $(0,0)$, $(2,0)$, $(0,3)$ and $(2,3)$

$D_f = f_{xx} f_{yy} - (f_{xy})^2$

$= (16x - 12x^2)(24y - 12y^2) - (0)^2$

$D_f(0,0) = D_f(2,0) = D_f(0,3) = 0$, test fails.

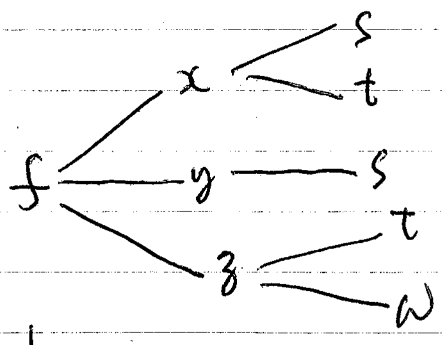
But at $(2,3)$, $f_{xx}(2,3) < 0$, $f_{yy} < 0$, $D_f(2,3) > 0$

$\therefore$ we have a local max at $(2,3)$ (which turns out to be an abs. max. as well).

Chain Rule for Partial Differentiation

Since we are dealing with functions of multi-variable, there is no single chain rule that could handle every possible case, but each individual case could be handled using a tree diagram.

Ex. Given $f(x,y,z)$ where x is a function of s & t, y a fct of s and z a function of t & w. As a result f is a function of s, t and w. The relationship could be expressed through a tree diagram.



we have $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{dy}{ds}$, $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t}$

$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$

Ex. $f(x,y,z) = xyz$
where $x = s^2 + t^2$, $y = st$, $z = tw^2$

Ex. Proof of the 2^{nd} derivative test

Let $f(x,y)$ have a critical pt. at (a,b) where f is twice continuous differentiable and $\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0$

Without loss of generality, we assume (a,b) is at $(0,0)$.

In order to show that f either attains a local max or a local min at (a,b), it boils down to showing that the curve of intersection between the surface $z = f(x,y)$ and ~~any~~ θ-plane (the vertical plane which makes an angle θ with the positive x-axis)

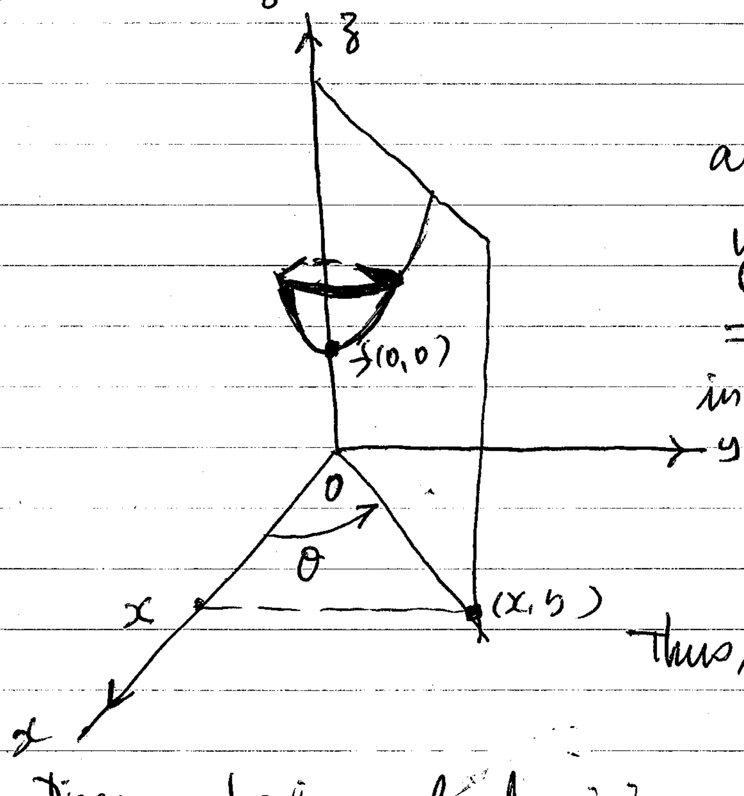


Diagram showing a local minimum at $(0,0)$

In this case, x and y are related through the eqn.

$y = \tan\theta\, x$

$\Rightarrow$ equation of curve of intersection is given by

$$z = f(x,y) = f(x, \tan\theta x)$$

Thus, f has the tree diagram

$$f \begin{cases} x \\ y - x \end{cases}$$

Indeed, set $g(x) = f(x, x\tan\theta)$, ← equation of curve of intersection between the surface & the θplane

$$g'(x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = f_x + f_y \tan\theta$$

Since $(0,0)$ is a critical point i.e. $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0)$, we must have

$$g'(0) = f_x(0,0) + f_y(0,0)\tan\theta = 0.$$

Consider now $g''(x) = f_{xx}(x, \tan\theta\, x) + 2\tan\theta\, f_{xy}(x, \tan\theta\, x) + f_{yy}(x, \tan\theta\, x)\tan^2\theta$

Here let us rule out the case $\theta = \pi/2$ & $3\pi/2$ (which actually corresponds to the case $x = 0$), so that $\tan\theta$ exists.

$$g''(0) = f_{xx}(0,0) + 2f_{xy}(0,0)\tan\theta + f_{yy}(0,0)\tan^2\theta$$

This is a quadratic function in $\tan\theta$ with discriminant

$$\Delta = 4f_{xy}(0,0)^2 - 4f_{xx}(0,0)f_{yy}(0,0)$$
$$= -4D_f(0,0)$$

Now that if $D_f(0,0) > 0$, $\Delta < 0 \Rightarrow g''(0) < 0$ or $g''(0) > 0$ for every $\theta \neq \pi/2$ and $3\pi/2$ and the sign of $g''(0)$ is determined by

$g''(0) > 0$ if $f_{yy}(0,0) > 0$ (alternatively $f_{xx}(0,0) > 0$)

$g''(0) < 0$ if $f_{yy}(0,0) > 0$ (alternatively $f_{xx}(0,0) < 0$)

Thus, $D_f(0,0) > 0$ and $f_{xx}(0,0) > 0 \Rightarrow$ a local minimum at $(0,0)$ in all θ-direction such that $\theta \neq \pi/2, 3\pi/2$

$D_f(0,0) > 0$ and $f_{xx}(0,0) < 0 \Rightarrow$ a local maximum at $(0,0)$ in all θ direction s.t. $\theta \neq \pi/2, 3\pi/2$

Finally, we consider the case when $\theta = \pi/2$ or $3\pi/2$, but this is the trivial case when $x = 0$ (i.e. the θ-plane co-incides with the y-z plane), we'll leave it as an exercise for the readers.

In the case when $D_f(0,0) < 0$, we have $g''(0) > 0$ along certain θ-directions & $g''(0) < 0$ along some other θ directions, this would correspond to the saddle point case //

Implicit Function Theorem

Th^m. Consider the equation $F(x,y,z)=0$ such that F is continuously differentiable in all its arguments. Suppose further that at some point (x_0,y_0,z_0), $\frac{\partial F}{\partial z}(x_0,y_0,z_0)\neq 0$, then $F(x,y,z)=0$ defines implicitly z as a function of x & y locally at (x_0,y_0) i.e. we could find a small neighborhood of (x_0,y_0) such that $z=h(x,y)$ satisfies

$$F(x,y,h(x,y))=0$$

Further, $z=h(x,y)$ is partially differentiable with respect to x & with respect to y, with

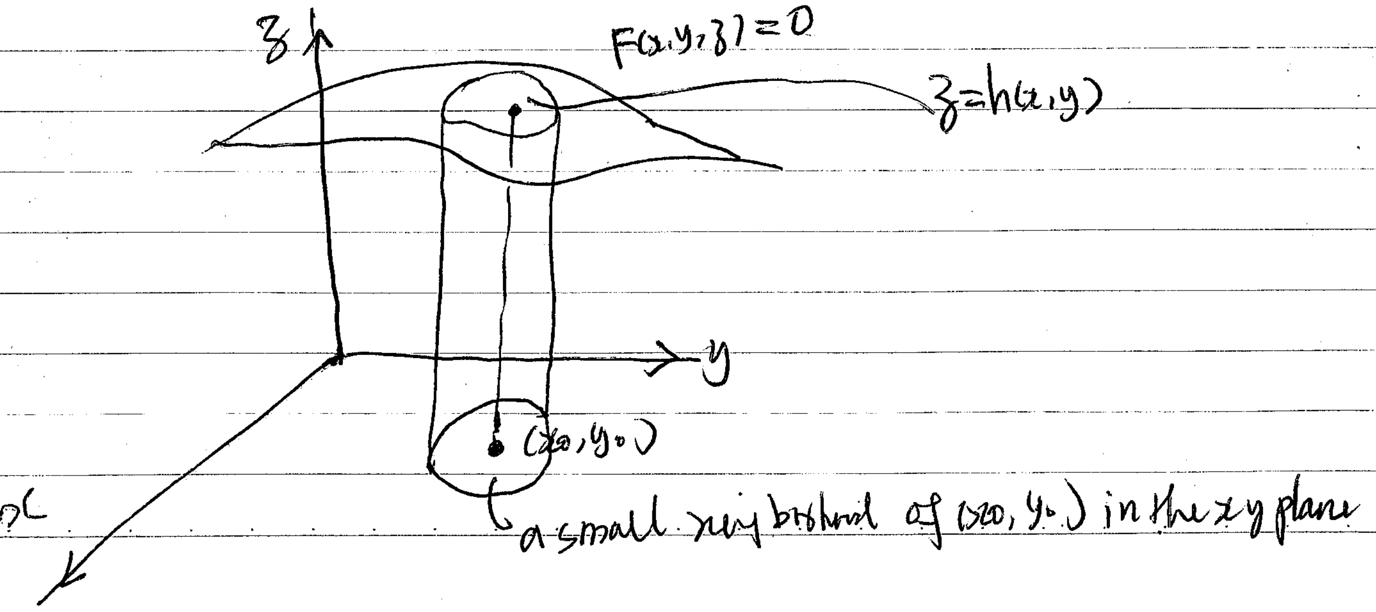
$$\frac{\partial z}{\partial x}(x_0,y_0)=\frac{\partial h}{\partial x}(x_0,y_0)=-\frac{\frac{\partial F}{\partial x}(x_0,y_0,z_0)}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)}$$

and

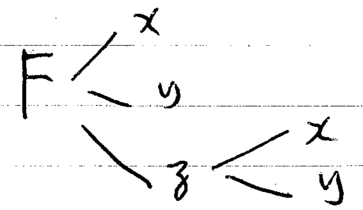
$$\frac{\partial z}{\partial y}(x_0,y_0)=\frac{\partial h}{\partial y}(x_0,y_0)=\frac{-\frac{\partial F}{\partial y}(x_0,y_0,z_0)}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)}$$

Idea of the Theorem:

Suppose indeed, $F(x,y,z)=0$ defines z implicitly as function of x & y locally at (x_0,y_0) (see picture below)



Nearby (x_0, y_0), we have $F(x, y, h(x, y)) \equiv 0$, we could look upon L.H.S. as a function of x & y. Thus,



Differentiate both sides partially with respect to x & y respectively,

$$\frac{\partial F}{\partial x}(x,y,z) + \frac{\partial F}{\partial z}(x,y,z)\frac{\partial h}{\partial x}(x,y) = 0$$

In particular, at (x_0, y_0, z_0), $\displaystyle \frac{\partial h}{\partial x}(x_0, y_0) = \frac{-\frac{\partial F}{\partial x}(x_0,y_0,z_0)}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)}$

$\therefore \displaystyle \frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ at (x_0, y_0).

$\displaystyle \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$ is proved analogously.

Remarks

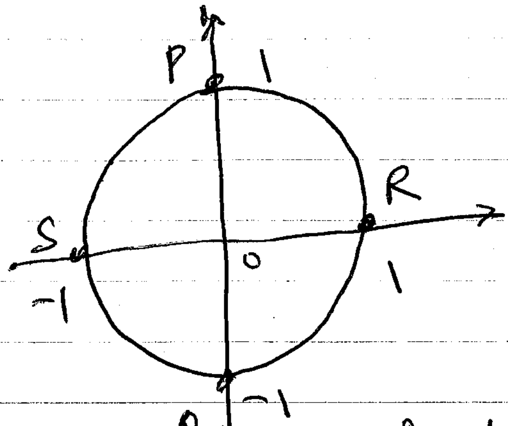
This result is not just confined to the case $F(x,y,z) = 0$, it applies to F which is a function of n variables for any $n \geq 2$

Ex. $F(x_1, x_2, \cdots, x_n) = 0$

$$\frac{\partial F}{\partial x_n} \neq 0 \Rightarrow \frac{\partial x_n}{\partial x_i} = -\frac{F_{x_i}}{F_{x_n}} \quad \forall\, i = 1 \cdots n-1$$

and so on.

Ex Consider $F(x,y) = x^2+y^2-1=0$, $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = 2y$



at $P(0,1)$ & $Q(0,-1)$, $\frac{\partial F}{\partial x} = 0$ but $\frac{\partial F}{\partial y} \neq 0$

$\therefore$ near by P & Q, we could solve for y as a fct of x. Indeed, we have

$y = \sqrt{1-x^2}$ nearby P & $y = -\sqrt{1-x^2}$ at Q.

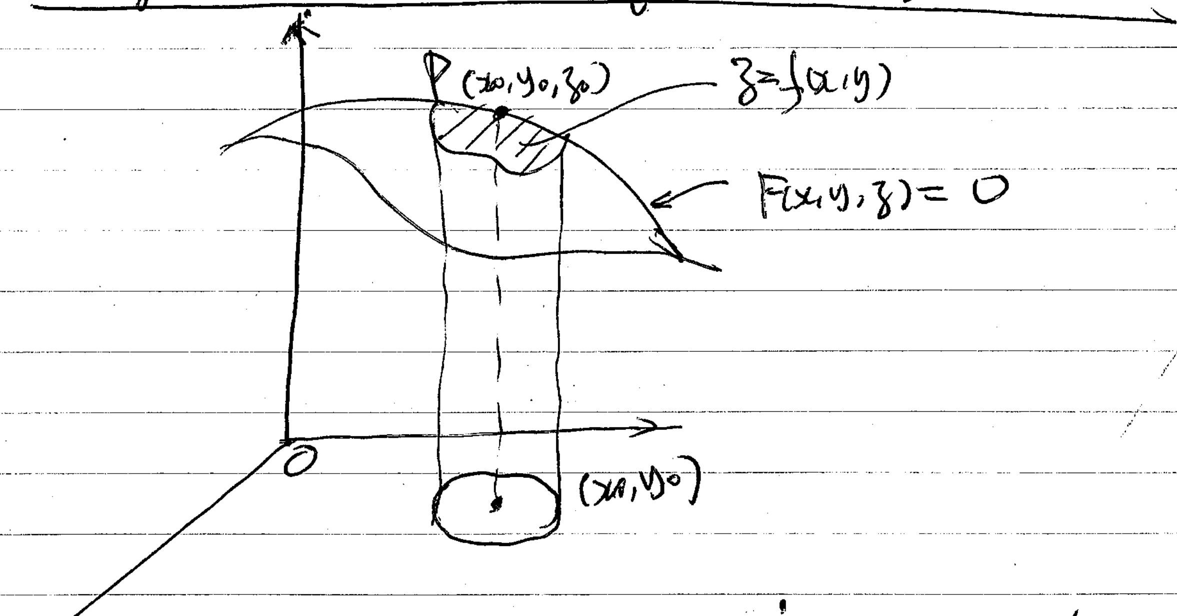
On the other hand, at $S(-1,0)$ & $R(1,0)$

$$\frac{\partial F}{\partial x} \neq 0$$

we could therefore solve of x as a fct of y nearby R & S.

i.e. we have $x = \sqrt{1-y^2}$ nearby R & $x = -\sqrt{1-y^2}$ nearby S.

Tangent Plane to the surface $F(x,y,z)=0$



Suppose one of the $\frac{\partial F}{\partial x}(P)$, $\frac{\partial F}{\partial y}(P)$ & $\frac{\partial F}{\partial z}(P) \neq 0$, without loss of generality, assume $\frac{\partial F}{\partial z}(x_0,y_0,z_0) = 0$

Then locally at P, we could have $z=f(x,y)$ s.t.

$$\frac{\partial f}{\partial x}(x_0,y_0) = -\frac{\frac{\partial F}{\partial x}(P)}{\frac{\partial F}{\partial z}(P)}, \quad \frac{\partial f}{\partial y} = -\frac{\frac{\partial F}{\partial y}(P)}{\frac{\partial F}{\partial z}(P)}$$

The normal vector N at P is given by

$$\vec{N} = \left\langle -\frac{\partial f}{\partial x}(x_0,y_0), -\frac{\partial f}{\partial y}(x_0,y_0), 1 \right\rangle$$

$$= \left\langle \frac{\frac{\partial F}{\partial x}(P)}{\frac{\partial F}{\partial z}(P)}, \frac{\frac{\partial F}{\partial y}(P)}{\frac{\partial F}{\partial z}(P)}, 1 \right\rangle$$

or we could simply take $\tilde{N} = \left\langle \frac{\partial F}{\partial x}(P), \frac{\partial F}{\partial y}(P), \frac{\partial F}{\partial z}(P) \right\rangle$.

We denote it as $\nabla F(P)$.

The other two possibilities of having $\frac{\partial F}{\partial x}(P) \neq 0$ or $\frac{\partial F}{\partial y}(P) \neq 0$

could be argued ~~in~~ exactly the same way.

Ex. Consider the surface $F(x,y,z) = 2z^3 + (x+y)z^2 + x^2 + y^2 = 14$, $P(2,2,1)$ is a point on the surface. We could take

$\vec{N} = \nabla F(2,2,1)$ as a normal vector at the pt. $(2,2,1)$ (note that whether we bring the 14 to the R.H.S. or not is irrelevant).

$$\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle = \left\langle z^2 + 2x, z^2 + 2y, 6z^2 + 2(x+y)z \right\rangle$$

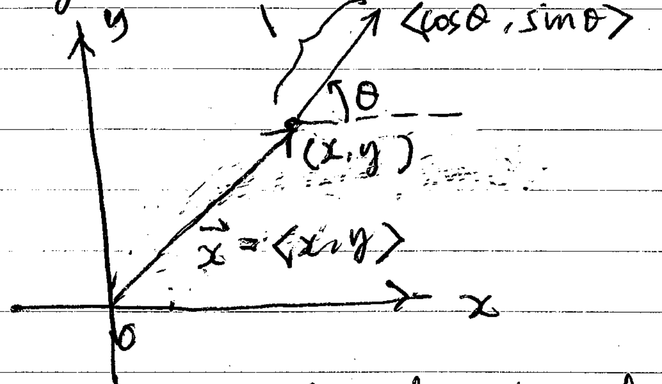
$$\Rightarrow \vec{N} = \nabla F(2,2,1) = \langle 5, 5, 14 \rangle$$

Equation of tangent plane is therefore,

$$5(x-2) + 5(y-2) + 14(z-1) = 0 \quad //.$$

Directional Derivatives of $f(x,y)$

This is a generalization of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for a function $f(x,y)$. Let now $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ be a direction vector in the xy plane (recall, a direction vector $\vec{u}$ is a vector pointing in a certain direction of length 1).



We define the directional derivative of f in the direction $\vec{u}$ as follows. We use vector notation with $\vec{x} = \langle x, y \rangle$ and $\vec{x} + h\vec{u} = \langle x + h\cos\theta, y + h\sin\theta \rangle$ and define

$$\frac{\partial f(\vec{x})}{\partial \vec{u}} = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

We construct the auxillary

$$g(t) = f(x + t\cos\theta, y + t\sin\theta)$$

then $g'(t) = \frac{\partial f}{\partial x}(x + t\cos\theta, y + t\sin\theta)\cos\theta + \frac{\partial f}{\partial y}(x + t\cos\theta, y + t\sin\theta)\sin\theta$

In particular, $g'(0) = \frac{\partial f}{\partial x}(x,y)\cos\theta + \frac{\partial f}{\partial y}(x,y)\sin\theta$

But $g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$ also,

thus $$\lim_{h \to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

<u>Remark</u>: Note that $\frac{\partial f}{\partial \vec{i}}(x,y) = \frac{\partial f}{\partial x}(x,y)$, $\frac{\partial f}{\partial \vec{j}}(x,y) = \frac{\partial f}{\partial y}(x,y)$

In general, suppose we have a function of n variables $f(x_1,\cdots,x_n)$, set $\vec{x}=\langle x_1,\cdots,x_n\rangle$ and let $\vec{u}=\langle u_1,\cdots,u_n\rangle$, $|\vec{u}|=1$ be any direction vector, we define

$$\frac{\partial f}{\partial \vec{u}}(\vec{x}) = \lim_{h\to 0}\frac{f(\vec{x}+h\vec{u})-f(\vec{x})}{h}$$

to be the directional derivative of $f(\vec{x})$ in the direction $\vec{u}$.

Definition: Given $f(\vec{x})=f(x_1,\cdots,x_n)$, $\vec{x}=(x_1,\cdots,x_n)$, we define the gradient vector of f by

$$\nabla f(\vec{x}) = \left\langle \frac{\partial f(x_1,\cdots,x_n)}{\partial x_1}, \frac{\partial f(x_1\cdots x_n)}{\partial x_2}, \cdots, \frac{\partial f(x_1,\cdots,x_n)}{\partial x_n}\right\rangle$$

Then along any direction $\vec{u}$, we have

$$\boxed{\frac{\partial f(\vec{x})}{\partial \vec{u}} = \nabla f(\vec{x})\cdot\vec{u} = \frac{\partial f}{\partial x_1}(x_1\cdots x_n)u_1 + \cdots + \frac{\partial f}{\partial x_n}(x_1\cdots x_n)u_n}$$

Remark: $\frac{\partial f}{\partial \vec{u}}$ is the rate of change of f in the direction of $\vec{u}$.

Ex. Given $f(x,y,z)=x^2+y^2+z^2-2xy+2xz+yz$, find the directional derivative of f at the point $P(1,0,2)$ in the direction from P to $Q(2,3,3)$

Solution:

$$\nabla f(x,y,z) = \langle 2x-2y+2z,\ 2y-2x+z,\ 2z+2x+y\rangle$$

$$\therefore \nabla f(1,0,2) = \langle 2+4,\ -2+2,\ 4+2\rangle = \langle 6,0,6\rangle$$

and the direction vector in the direction of $\vec{PQ}=\langle 1,3,1\rangle$ is given by $\vec{u}=\frac{\langle 1,3,1\rangle}{\sqrt{1+9+1}} = \left\langle \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right\rangle$

$$\therefore \frac{\partial f}{\partial \vec{u}}(1,0,2) = \nabla f(1,0,2)\cdot\vec{u} = \langle 6,0,6\rangle\cdot\left\langle\frac{1}{\sqrt{11}},\frac{3}{\sqrt{11}},\frac{1}{\sqrt{11}}\right\rangle = \frac{12}{\sqrt{11}}$$

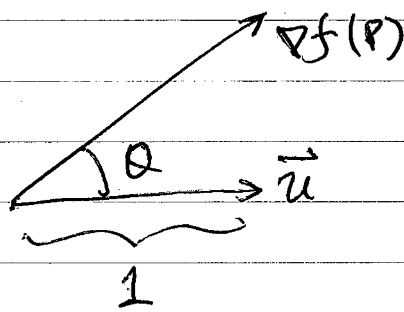
Direction in which f changes most rapidly and the Direction in which the rate of change of f is a minimum.

Let $P(x_0, y_0, z_0)$ be any point belonging to the domain of f and let $\vec{u}$ be any direction, then

$$\frac{\partial f}{\partial \vec{u}}(P) = \nabla f(P) \cdot \vec{u}$$

By dot product's property, $\frac{\partial f}{\partial u}(P) = |\nabla f(P)||\vec{u}|\cos\theta = |\nabla f(P)|\cos\theta$

where θ is the $\measuredangle$ between $\nabla f(P)$ and $\vec{u}$



We observe that, $\frac{\partial f}{\partial u}(P)$ attains its maximum value of $|\nabla f(P)|$ when $\theta = 0$, i.e. when $\vec{u}$ is in the same direction of $\nabla f(P)$.

Similarly, the rate of change of f i.e. $\frac{\partial f}{\partial u}(P)$ attains its minimum value of $-|\nabla f(P)|$ when $\vec{u}$ is in the opposite direction of $-\nabla f(P)$.

Ex Suppose the temperature T (in °C) at the pt. (x, y, z) in space is given by $T(x,y,z) = x^3 - xy^2 - z$, in which direction does the temperature increase most rapidly at $P(1,1,0)$? What is the maximal rate of change at $P(1,1,0)$?

Solution: $\nabla T(x,y,z) = \langle 3x^2 - y^2, -2xy, -1 \rangle$

at $P(1,1,0)$, maximum rate of change of T is in the direction of $\nabla T(1,1,0) = \langle 2, -2, -1 \rangle$ or $\vec{u} = \langle \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \rangle$ with max rate of change $= |\nabla T(1,1,0)| = 3$//